GROWTH CYCLES IN A DISCRETE, NONLINEAR MODEL

by

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Abstract: This paper develops a discrete, nonlinear growth cycle model for a macroeconomy. The nonlinearities, which correspond to empirical relationships between profitability and capacity utilization in the postwar U.S **economy, can produce stable, periodic and chaotic behavior. These behaviors are** established analytically, and **further investigated through simulation. Data from the simulations are used to show that chaotic attractors can produce time series which are useful representations of business cycles.**

1. INTRODUCTION'

Goodwin's original (1967) work on growth cycles has stimulated much subsequent research. Since his model assumed full utilization of capital and generates rather long cycles, some effort has been put into examining the possibilities of self-sustained growth oscillations when capacity utilization is variable (Desai, 1973; Foley, 1987; Jarsulic, 1988, pp. 47-65; Medio ,1980; Semmler ,1987; Skott, 1989). Many of these models assume nonlinear investment functions, which are the source of the interesting dynamics.

The models often are set up as two-dimensional continuous systems, which allow use of the Poincare-Bendixson or Hopf bifurcation theorems to establish conditions for the existence of cycles. Such a choice of modeling technique of course restricts the possible dynamics, since chaos cannot occur in continuous systems of dimension less than three.

The model developed in subsequent sections introduces some new elements to this line of investigation. First, the dynamics of the model depend on nonlinearities related to income distribution. They derive from connections between potential profltability and capacity utllization, which play a large part in some current neo-marxian macro literature (Bowles et al., 1989). Hence it develops an empirically relevant case where the interesting dynamics are not derived exclusively from the aggregate demand side. Second, the model is cast in difference rather than differential equation form. It is shown analytically that stable, periodic and chaotic behavior are all possible in one and two dimensions. In this way the menu of dynamical alternatives is lengthened. Now of course what is on the menu is not always available in the restaurant. indeed, the analytical results establish the possibilfty of chaos, but not the existence of chaotic attractors of positive measure. Therefore computer simulation is used to show that chaos occurs in a detectable way for some, but not all, configurations of the model. The data generated by these simulations are also interesting because, like actual business cycle time series, they can produce runs of increasing, decreasing, or relatively stable data before exhibiting a marked change in direction. This indicates that chaotic, dynamical systems may be more suited to business cycle modeling than is sometimes supposed.

2. A DISCRETE GROWTH CYCLE MODEL

At the center of the model is the relationship between current profitability and the rate of capacity utilization. It is drawn from, and is a drastic simplification and compression of, ideas which are summarized in Bowles et al. (1989). There they suggest that, ceteris paribus, the rate of profit first rises, then begins-to decline, as the rate of capacity utilization increases. The theoretical explanation for this phenomenon lies in the marxian idea that there is conflict between workers and employers over the payment for labor time and the intensity of work. At lower levels of utilization workers' position is weaker. Therefore they can be made to work harder and real wage growth can be more easily restrained. As utilization increases, their power becomes greater relative to employers, and real wages can rise more rapidly and/or work effort can decline. These relationships can be summarized in the following equations:

$$\pi_t = F(u_t, \omega), \quad F'_1 > 0,\ F'_2 < 0 \tag{1}$$

$$\omega = G(u_t), \quad G' > 0 \tag{2}$$

$$u_t = Y_t/K_t \tag{3}$$

Equation (1) is a general functional form for the relationship between the rate of profit, π, and the rate of capacity utilzation, u. If there were no changes in the real wage or the intensity of work, profits would increase with utilization. However, real wages and work intensity, which are reflected in labor's share ω, are assumed to affected by current rates of utilization as in equatlon (2). The variable **u** is deflned in (3) as Y/K, NNP divided by the real capital stock. The use of Y/K as a measure of relative power is based on the assumptions that the labor force is a socially defined part of the population which moves more of less in step with the size of productive capital; and that employment is a proportional (or more than proportional, when work intensity is declining) function of output. It is not a bad empirical assumption for the post war US economy, as the discussion in Bowles et al. (1989, pp. 126-28) indicates.

To make the elements of (1)-(3) tractable for dynamical analysis we want to represent them as a piecewise linear function

$$\pi_t = \begin{cases} Au_t & 0 < u_t \leq u^* \\ B - Cu_t & u^* \leq u_t \leq u_{max} \end{cases} \tag{4}$$

Next we turn to aggregate demand and goods market clearing. We will make the assumption that depreciated capital is replaced, and that net investment is determined by profits, with certain lags. The time delay is attributable to order-construction lags. Thus investment will reflect the Kaleckian/neo-marxian belief that profits are the central determinant of investment expenditure. (See, for example, Kalecki , 1971; Marglin 1985, pp. 52-95). This belief receives empirical support in the neo-Keynesian (Fazzari et al., 1988) and neo-marxian (Bowles et al., 1989; Gordon et al., 1988) literature. The rate of growth of the capital stock is given by

$$g_t = \pi_{t-1} \tag{5a}$$

or

$$g_t = (\pi_t + \pi_{t-1})/2. \tag{5b}$$

where $g_t = [K_t/K_{t-1}] - 1$. On the consumption side we will assume a Kaldorian consumption function, where a constant proportion of wages, c_w, and a constant proportion of profits, c_π, are consumed. Given the inequalities $1 > c_w > c_\pi > 0$, and assuming that the goods market clears[2], we have the following equilibrium condition for the goods market[3]

$$u_t = ag_t - b\pi_t \tag{6}$$

where $a = 1/(1-c_w)$, $b = (c_w-c_\pi)/(1-c_w)$.

3. THE DYNAMICS OF THE MODEL

We can now consider two cases of this model. To construct the first we will use (4), (5a), and (6), which can be combined to produce a first order difference equation of the form

$$\pi_t = \begin{cases} \gamma\pi_{t-1} & 0 \le \pi_t \le \pi^* \\ \varphi - \phi\pi_{t-1} & \pi^* \le \pi_t \le \pi_{max} \end{cases} \qquad (7)$$

$\gamma = Aa/(1+Ab)$, $\varphi = B/(1-Cb)$, $\phi = C/(1-Cb)$. To make (7) easier to analyze, it will be made differentiable by assuming smoothness in an arbitrarily small neighborhood around the peak. That is (7) will be replaced by

$$\pi_t = \begin{cases} \gamma\pi_{t-1} & 0 \le \pi_t \le \pi^*-\varepsilon \\ f(\pi_{t-1}) & \pi^*-\varepsilon \le \pi_{t-1} \le \pi^*+\varepsilon \\ \varphi - \phi\pi_{t-1} & \pi^*+\varepsilon \le \pi_t \le \pi_{max} \end{cases} \qquad (8)$$

where f is a smooth function with $f' = \gamma$ at $\pi^*-\varepsilon$, $f' = -\phi$ at $\pi^*+\varepsilon$, where $\varepsilon > 0$. Equation (8) is represented graphically in Figure 1. A condition sufficient to keep this system inside its defined range is $\gamma < 1 + \gamma/\phi$, which will be assumed.

Note that the slope of the downward sloping segment in Figure 1 is determined by the coefficient ϕ. This is turn is determined by the coefficients C and b. C reflects the tendency of profitability to fall as utilization increases -- the more rapidly profits fall, the bigger is C and hence ϕ. Increases in c_π will reduce the absolute value of b, while increases in c_W will cause it to decrease. Therefore the bigger the multiplier effects of shifts in income distribution, the larger this coefficient.

There are two equilibrium points in this system, labeled and P_0 and P_1 in the diagram. Their stability properties will depend on the eigenvalues of (8) at those points. At P_0, it is given by $d\pi_t/d\pi_{t-1} = \gamma$, and at P_1 by $d\pi_t/d\pi_{t-1} = -\phi$ (Devaney, 1986, pp. 170-74). Hence P_0 will be unstable if $\gamma > 1$, which will be true if c_W and c_π are large enough; and P_1 will be unstable if $|\phi| > 1$, which will be true if the tendency of profits to decline is strong, or if distribution-related demand effects are large. The most interesting configuration of this system occurs when both the fixed points are unstable. Then it is possible to establish analytically the possibility of a wide variety of dynamics.

To investigate the dynamics of (8), one can use a theorem which establishes sufficent conditions for chaos in n-dimensional difference equations. First we need

Definition 1: Let $x_{k+1} = F(x_k)$, where $x_k \in R^n$ and $F: R^n \to R^n$ is continuous. Also let F be differentiable in some closed n-dimensional ball $B_r(z)$. The point $z \in R^n$ is an expanding fixed point of F in $B_r(z)$ if $F(z) = z$ and all eigenvalues of the Jacobian $DF(x)$ exceed 1 in norm for all $x \in B_r(z)$.

Definition 2: Assume that z is an expanding fixed point of F in $B_r(z)$ for some $r > 0$. Then z is said to be a snao-back repeller of F if there exists a point x_0 in $B_r(z)$ such that $x_0 \neq z$, $F^m(x_0) = z$, and $|DF^m(x_0)| \neq 0$ for some positive integer $m > 1$. Here F^m represents the composition of F with itself m times.

The following can be shown:

Theorem: If **F is differentiable and has a snap-back** repeller, then **F is** chaotic.4

Proof: Marrotto (1978).

Then it is possible to establish the following:

Proposition 1: If $\phi, \gamma > 1$, $1 + \gamma/\phi > \gamma$, and $([1+\gamma/\phi]/2) < \gamma^2$, then (8) **is chaotic.**

Proof: Consider Figure 2. Points π_0 and π_1 **always exist. Now if it is the case that** $\pi_2 < \gamma\pi^*$, **then there exists an** π_3 **such that** $\gamma\pi^* > \pi_3$, $\pi_0 > \pi_3$, **and for which** $F^3(\pi_3) = \pi_0$. **The condition** $\pi_2 < \gamma\pi^*$ **will be met if** $(1 + \gamma/\phi)(1 - 1/(1+\phi)) < \gamma^2$. Since $\gamma > 1$, it is sufficient to **have** $([1+\gamma/\phi]/2) < \gamma^2$ **to ensure that** $\pi_2 < \gamma\pi^*$.

Define $r = \pi_0 - \pi_3$, $B_r(\pi_0) = [q\text{-}r, \pi_0 + r]$, **and note that** $\pi_0 + r < (1 + \gamma/\phi)\pi^*$. **We know that** $F'(\pi) = -\phi$ **for all** $\pi \in B_r(\pi_0)$. **This makes** π_0 **an expanding fixed point. Since** $|DF^3(\pi_3)| = [F'(F^2(\pi_3))][F'(F^1(\pi_3))][F'(\pi_3)] \neq 0$, π_0 is a snap-back repeller.

While this proposition establishes the possibility of a wide variety of dynamical behavior for (8), it does not prove that the chaotic set will have positive measure. **This previously has** been recognized as limiting the usefulness of existence results (Day and Schaefer, 1935; Baumol and Benhabib, 1989). It is, however, possible to use computer simulation techniques to learn more about the system. Proposition 1 suggests the possiblity of very complex periodic behavior, and that indeed manifests itself for different combinations of parameter values. Setting $\gamma = 1.1$, $\phi =$

1.05, constructing a simple function4 for $f(\pi_{t-1})$, and then simulating the system for 25 iterations produces the time series plot of Figure 3. This appears to be a 4-cycle. Raising the value of ϕ to 2.1 produces the far more complex time series of Figure 4. The histogram for 5000 interations of this series, in Figure 5, looks stochastic.

To test for aperiodicity, one can try to establish the dimension of an attractor empirically. Chaotic attractors often are characterized by fractal, i.e. non-integer, Hausdorff dimensions (Berge et al., 1934, pp. 112-14). The Hausdorff dimension is defined by looking at the minimum number of n-dimensional hypercubes which cover the attractor in n-space. When M(k), the minimal number of hypercubes of length L, is related to L by the approximation

$$M(L) = L^{-D} \quad (9)$$
$$L \to 0$$

then the dimension is D. For a chaotic attractor in l-space, D will be greater than zero but less than 1. In a series of papers, Grassberger and Procaccia (1983a, 1983b, 1984) have established a method for estimating dimension. They show that the correlation intergral, defined by

$$C(k) = \lim_{N \to \infty} \sum_{i,j} H(k, Z_{ij}) / (N(N-1)) \quad (10)$$

where N is the number of points, Z_{ij} is the distance between points i and j, and H is the Heaviside function defined by

$$H(k,Z) = \begin{cases} 0, & k < Z \\ 1, & k > Z \end{cases} \qquad (11)$$

is characterized by

$$C(k) \sim k^{\gamma} \qquad (12)$$

They also show that the value of the coefficient y provides a lower bound to the Hausdorff dimension of an attractor. Furthermore, the value of y often turns out to provide, in practice, a good estimate of D.

The Grassberger-Procacci a method can be implemented (Grassberger and Procaccia, 1983b, Schaffer et al., 1986) by constructing successively higher level Takens (1981) embeddings of a single time series. A n-dimensional Takens embedding is constructed by taking the observations on a variable x and creating an n-dimensional vector $z_i = (x(t_i), x(t_i + q) \ldots x(t_i + [n-1]q))$, where t_i and q are time indices. Then **for each level of embedding,** $\Delta \ln C(k)/\Delta \ln(k)$ is plotted against $\ln(k)$ for a set of values of $\ln(k)$. From this plot, a Set of values of $\ln(k)$ where $\Delta \ln C(k)/\Delta \ln(k)$ is stable is selected, and a least squares regression of $\ln C(k)$ against $\ln(k)$ is calculated. The slope of the regression is the estimate of v from expression (12). if the values of Y from the regressions appear to converge as the embedding dimension rises, then they are taken as an estimate of the Hausdorff dimension

The correlation integrals for (8), with $\gamma = 1.1$ and $\phi = 2.1$, and with Takens embeddings of one through four, were calculated for a simulation of 1000 iterates. These are displayed in Figure 6, and the statistical results of the regressions are given in Table 1. The results *show* that the estimate of dimension is not statistically different from one. Similar results occur for various values of ϕ and γ ,and for simulations of different length. We conclude that the chaotic set for (8) is difficult to identify. Since actual economic time series are usually shorter than those used here, it would undoubtedly be difficult to identify chaotic behavior if the underlying economic process were similar.

To see that aperiodicity is more easily identifiable in a system which is only slightly different, we can consider a second case. Using (4),(6) and (5b), which allows for longer lag effects in capital accumulation, we have

$$\pi_t = \begin{cases} \gamma(\pi_{t-1} + y_{t-1})/2 & 0 \le \pi_{t-1} + y_{t-1} \le \pi^* - \varepsilon \\ h(\pi_{t-1} + y_{t-1}) & \pi^* - \varepsilon \le \pi_{t-1} + y_{t-1} \le \pi^* + \varepsilon \\ \phi - \phi(\pi_{t-1} + y_{t-1})/2 & \pi^* + \varepsilon \le \pi_{t-1} + y_{t-1} \le \pi_{max} \end{cases} \tag{13}$$

$$y_t = \pi_{t-1}$$

where $\varepsilon > 0$ and where h is a smooth function with $h' = \gamma/2$ at $\pi^* - \varepsilon$ and $h' = -\phi/2$ at $\pi^* + \varepsilon$. The system (13) has a non-zero fixed point at $\pi_0 = \phi/(1+\phi) = (\phi + \gamma)\pi^*/(1+\phi)$, $y_0 = \pi_0$. *We* again need to assume that $(1 + \gamma/\phi) > \gamma$ to prevent the the system from expanding out of range. Constructing the Jacobian of (13) snd using it to solve for the eigenvalues λ of the system gives A = $[1/2][-\phi/2 \pm$

$[(\phi/2)^2 - 2\phi]^{1/2}]$. To have the fixed point be unstable but not a saddle point (i.e. to have it an expanding point), we need $\mathrm{mod}(\lambda) > 1$.

Marroto's theorem again can be used to learn more about the dynamics of (13).

Proposition 2: When $8 > \phi > 2$, $(1/2[\phi^2] - \phi) > 1$, $(1 + \gamma/\phi) > \gamma$, and γ is larger than but sufficiently close to 1, (13) is chaotic.

Proof: Consider the sequence of points in Table 2. By starting at the point (π_3, y_3) and applying (13), one Iterates up the values in the table to the fixed point (π_0, y_0). Now the distance between (π_3, y_3) and the fixed point is given by $d_3 = |4(\gamma-1)/\phi\gamma|\pi_0$. When d_3 is sufficiently small, all points in a two dimensional ball of radius d_3 centered at (π_0, y_0) will produce eigenvalues for the Jacobian of (13) given by $\lambda = [1/2][-\phi/2 \pm [(\phi/2)^2 - 2\phi]^{1/2}]$. For $8 > \phi > 2$, the eigenvalues will be complex. For values of ϕ satisfying $(1/2[\phi^2] - \phi) > 1$, $\mathrm{mod}(\lambda) > 1$ and all points in the ball will be expanding. Application of the chain rule implies a non-zero value for $|DF^3(\pi_3, y_3)|$, since $|DF| \neq 0$ everywhere along the path from (π_3, y_3). Hence (π_0, y_0) is a snap-back repeller.

When (13) is simulated for the parameter values $\gamma = 1.1$, $\phi = 3.5$, which according to proposition 2 gives (13) a chaotic attractor, an erratic time series is produced. **The statistical estimates of the dimension are given in** Table 3, and the scaling region is displayed graphically in Figure 7. The estimates appear to converge to a value of about around 1.5. In' this case the existence of a chaotic attractor is indicated. It is important to notice that this simulation produces series in which data increase, decrease or remain relatively stable for multiple periods. **This is** illustrated in Figure

8, in which forty points from the simulation are graphed. The values graphed there are listed in Table 4. Notice that these data imply an even smoother series for the rate of capital stock growth, which is a two period moving average of profit rates. This suggests that one commonly perceived shortcoming of chaotic systems, i.e. that they produce time series which are saw-toothed and therefore not adequate to represent the smoother time series of actual business cycles (e.g. Gabisch and Lorenz, 1989, p.189), does not apply to all chaotic systems. While no-one would pretend that the, simulated time series looks exactly like a representative macroeconomic series, it does have some family resemblance. This is not a bad outcome from a small, nearly piecewise linear system. It might be the case that a higher-dimension chaotic process can generate endogenously both the sustained movements and the stochastic blips one associates with business cycle data.

4. Conclusions

The model developed in this paper investigates the potential contribution of a nonlinearity in the profit-utilization relationship to growth dynamics. Using a discrete, nearly piece-wise linear framework, it has been shown that those contributions can be significant and varied. When the fixed points of the system are unstable, complex dynamics are possible. They can be periodic and aperiodic. When aperiodic, the models need not exhibit the alternating increases and decreases in value that are sometimes associated with chaotic systems. Runs of data in one direction or another are possible. In this respect, the chaotic model developed is qualitatively like actual business cycles, where turning points do not follow immediately on one another.

It has been emphasized that these dynamics are tied closely to profitabilty. The stability conditions of the model are determined by the strength of profit declines induced by utilization, and the strength of changes in consumption demand induced by changes in profitability. Moreover, profitability governs the rate of accumulation. The model thus echoes the connections drawn in classical economics between profitability and cycles, although it does so while taking account of utilization.

The discussion also illustrates the value of simulatlon when applying nonlinear dynamics to particular economic models. Definite analytical results are not easy to obtain for nonlinear models . However, the use of analytical techniques where possible, combined with data analysis of simulated models, can confirm (or frustrate) intuitions in a helpful way.

FOOTNOTE5

1. The support of the Jerome Levy Economics institute, Bard College, is gratefully acknowledged. The views expressed are those of the author and do not necessarily reflect the views of the Levy Institute.

2. The restriction that workers save something is necessary to rule out multiple solutions in this formulation. For the postwar US economy this is not a bad assumption (Marglin,1985, pp. 393-455). To get around the multiple solution case assuming no worker savings, one could recast aggregate demand relations along the lines of Foley (1987), or assume Robertsonian consumption relations, where current consumption is partly a function of previous income.

3. As is made clear in the text, the set-up of this model is derived directly from the work of Gordon, Bowles and Weisskopf. In fact the graphical framework, which inspired the dynamical analysis, was suggested by David Gordon's informal presentation of the theory underlying his recent econometric work. However, while my debt to these economists could not be more evident, they are not implicated in the analysis of this paper.

4. To say that F is chaotic means that there exists

(i) a positive integer N such that for each integer $p > N$, F has a point of period p;

(ii) a "scrambled set' of?; i.e. an uncountable set S containing no periodic points of F such that

(a) $S \supset F[S]$

(b) for every $X, Y \in S$ with $X \neq Y$

$$\limsup_{k \to \infty} \|F^k(X) - F^k(Y)\| > 0$$

(c) for every $X \in S$ and any periodic point Y of F

$$\limsup_{k \to \infty} \|F^k(X) - F^k(Y)\| > 0$$

(d) an uncountable set S_0 of S such that for every $X, Y \in S_0$

$$\liminf_{k \to \infty} \|F^k(X) - F^k(Y)\| = 0$$

4. In simulating this version of the model, the function connecting the linear segments was constructed by inscribing a circle of small radius tangent to a point on $\gamma\pi_{i-1}$. A straight line, with slope $-\phi$, tangent to the circle, determines the rest of the function F. While the result is differentiable, it is not much different from the piecewise-linear model. The same technique was used in the simulation of the two dimensional model.

BIBLIOGRAPHY

Baumol, W. and Behabi b, J. 1989. Chaos: Significance, Mechanism and Economic Applications. Journal of Economic Perspectives. Volume 3, pp. 77-106.

Berge, P., Pomeau, Y. and Vidal, C. 1984. Order Within Chaos. New York: J. Wiley and Sons.

Bowles, S., Gordon, D. and Weisskopf, T. 1989. Business Ascendancy and Economic Impasse. Journal of Economic Perspectives. Volume 3, pp. 107-34.

Day, R. and Schafer, W. 1985 Keynesian Chaos. Journal of Macroeconomics. Volume 7, pp. 277-295.

Devaney, R. 1986. An Introduction to Chaotic Dynamical Systems. Menlo Park, CA: Benjamin/Cummings Publishing Company.

Desai, M. 1973. Growth Cycles and Inflation in a Model of the Class Struggle. Journal of Economic Theory. Volume 6, pp. 527-45.

Fazzari, S., Hubbard, R., and Petersen, B. 1988. Financing Constraints and Corporate Investment. Brookings Papers on Economic Activity, No. 1, pp. 141-206.

Foley, D. 1987. Liquidity-Profit Rate Cycles in a Capitalist Economy. Journal of Economic Behavior and Organization. Volume 8, pp. 449-68.

Gabisch, G. and Lorenz, H.-W. 1989. Business Cycle Theory. New York: Springer-Verlag.

Goodwin, R. 1967. A Growth Cycle. In C. H. Feinstein (ed.), Socialism, Capitalism and Economic Growth. Cambridge: Cambridge University Press.

Gordon, D., Weisskopf, T. and Bowles, S. 1988. A Conflict Model of Investment: The Social Determinants of U. S. Capital Accumulation. Working Paper No. 2, Department of Economics, New School for Social Research.

Grassberger, P. and Procaccia, I. 1983a. Characterization of Strange Attractors. Physical Review Letters, Volume 50, pp. 346-49.

Grassberger, P. and Procaccia, I. 1983b. Measuring the Strangeness of Strange Attractors. Physica, 9D, pp. 189-208.

Grassberger, P. and Procaccia, I. 1984. Dimensions and Entropies of Strange Attractors From a Fluctuating Dynamics Approach. Physica, 13D, pp. 34-54.

Jarsulic, M. 1988. Effective Demand and Income Distribution. Cambridge (UK): Polity Press; and Boulder, Colorado: Westview Press.

Kalecki, M. 1971. Selected Essaus on the Dynamics of the Capitalist Economy. Cambridge: Cambridge University Press.

tlarglin, S. 1985. Growth, Distribution and Prices. Cambridge: Harvard University Press.

Medio, A. 198 1. A Classical Model of Business Cycles. In E. Nell (ed.), Growth Profits and Property. Cambridge: Cambridge University Press.

Schaffer, W., Ellner, S. and Kot, M. 1986. Effects of Noise on Some Dynamical Models in Ecology. Journal of Mathematical Biology. Volume 24, pp. 479-523.

Semmler, W. 1987. A Macroeconomic Limit Cycle With Financial Perturbations. Journal of Economic Rehavior and Organization. Volume 8, pp. 469-95.

Skott, P. 1989. Effective Demand, Class Struggle and Cyclical Growth. International Economic Review, Volume 30, pp. 231-47.

Takens, F. 198 1. Detecting Strange Attractors in Turbulence. In D. Rand and L.-S. Young (eds.) Dynamical Systems and Turbulence, Warwick 1980. New York: Springer-Verlag.

TABLE 1

n	a0	a1	ci	df	ln(k0)	ln(k1)
1	4.47	.98	.006	4	-9.5	-7.0
2	3.88	.99	..003	4	-9.5	-7.0
3.	3.47	1.00	.009	4	-9.5	-7.0
4	3.13	1.02	.02	4	-9.5	-7.0

n: embedding dimension; a_0: intercept of regression; a_1: slope of ragression and estimate of dimension; ci: 95 per cent confidence interval for a_1; df:. degrees of freedom for the regression coefficient a_1; k_0: the minimum value fork in the scaling region; k_1: the maximum value for k in the scaling region. The estimates were made using a simulation of 1,000 observations.

Table 2

Points in π_t, y_t space

$(\pi_0, y_0) = (\pi_0, \pi_0)$

$(\pi_1, y_1) = (\pi_0, [(2/\gamma) - 1]\pi_0)$

$(\pi_2, y_2) = ([(2/\gamma) - 1]\pi_0, \pi_0)$

$(\pi_3, y_3) = (\pi_0, [(4(\gamma - 1)/\phi\gamma) + 1]\pi_0)$

TABLE 3

n	a0	a1	ci	df	ln(k0)	ln(k1)
1	5.16	.98	.004	5	-10.5	-7.5
2	7.02	1.50	.025	5	-9.0	-6.0
3	6.39	1.49	.018	5	-8.5	-5.5
4	6.10	1.49	.022	6	-9.0	-5.5

n: embedding dimension; a_0: intercept of regression; a_1: slope of regression and estimate of dimension; ci: 95 per cent confidence interval for a_1; df: degrees of freedom for the regression coefficient a_1; k_0: the minimum value fork in the scaling region; k_1: the maximum value fork in the scaling region. The estimates were made using a sample of 1,000 observations.

Table 4

5.425598E-002
4.973886E-002
4.801036E-002
5.376207E-002
5.189957E-002
4.509346E-002
5.334616E-002
5.414179E-002
4.189740E-002
5.282156E-002
5.209543E-002
4.639660E-002
5.417061E-002
5.400870E-002
4.068752E-002
5.208292E-002
5.102374E-002
4.956466E-002
5.397161E-002
4.881284E-002
5.012853E-002
5.441776E-002
4.704532E-002
5 244095E-002
5.471745E-002
4.247413E-002
5.345537E-002
5.276122E-002
4.412229E-002
5.328593E-002
5.357452E-002
4.299552E-002
5.311352E-002
5.285998E-002
4.454770E-002
5.357422E-002
5.396705E-002
4.180410E-002
5.267413E-002
5.196303E-002

Simulation of (13), with $\gamma = 1.1$, $\varphi = 3.5$, 40 iterations

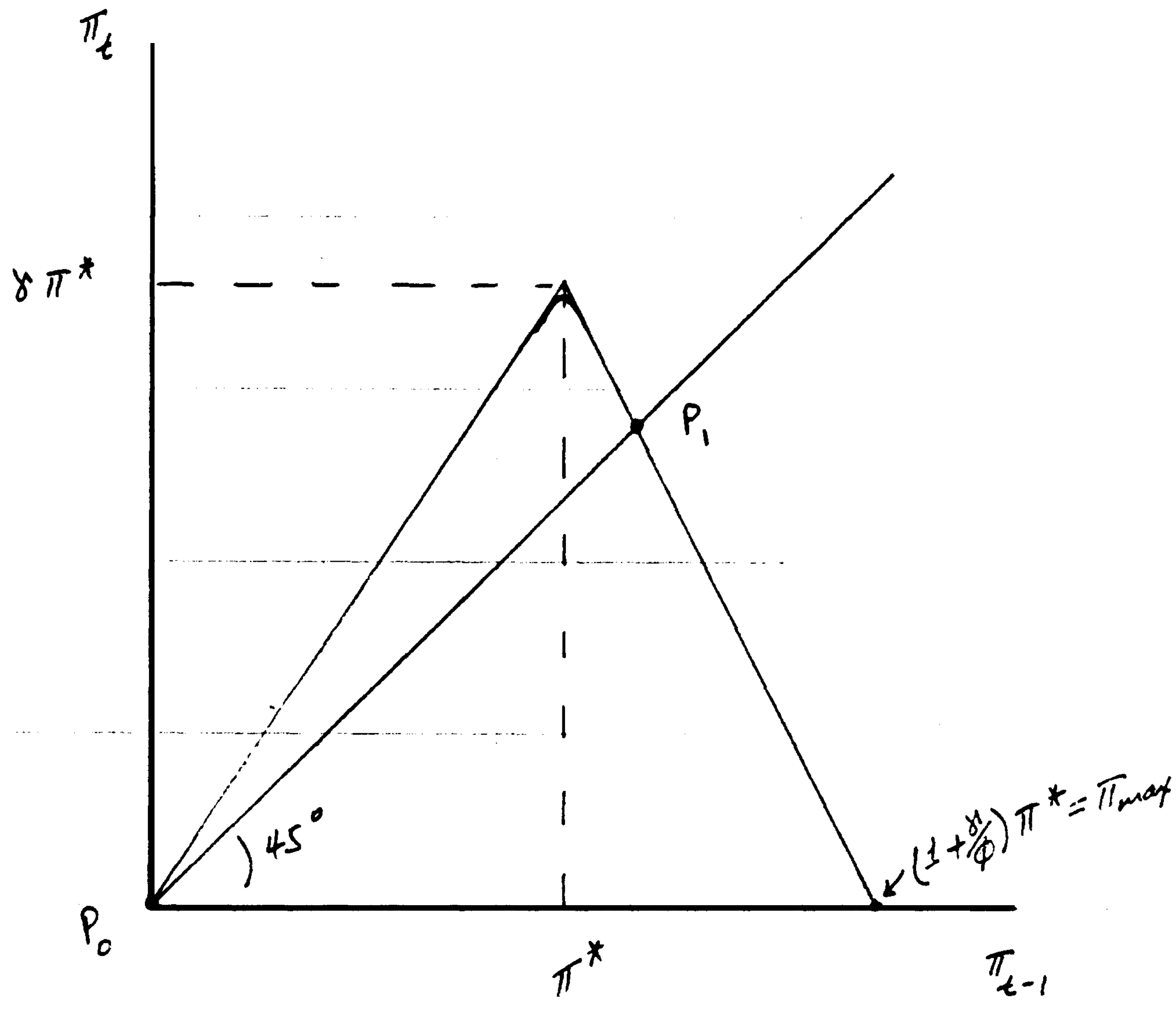


FIGURE 1

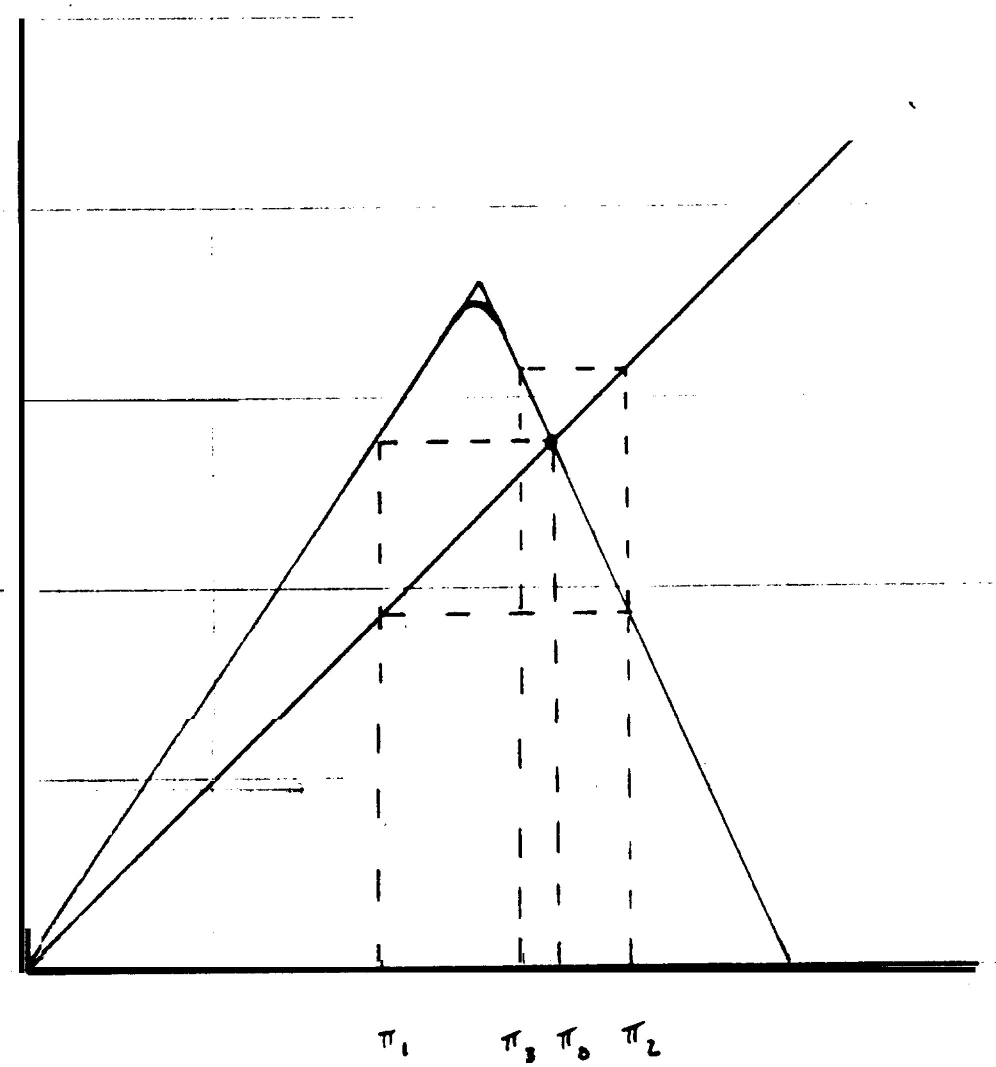
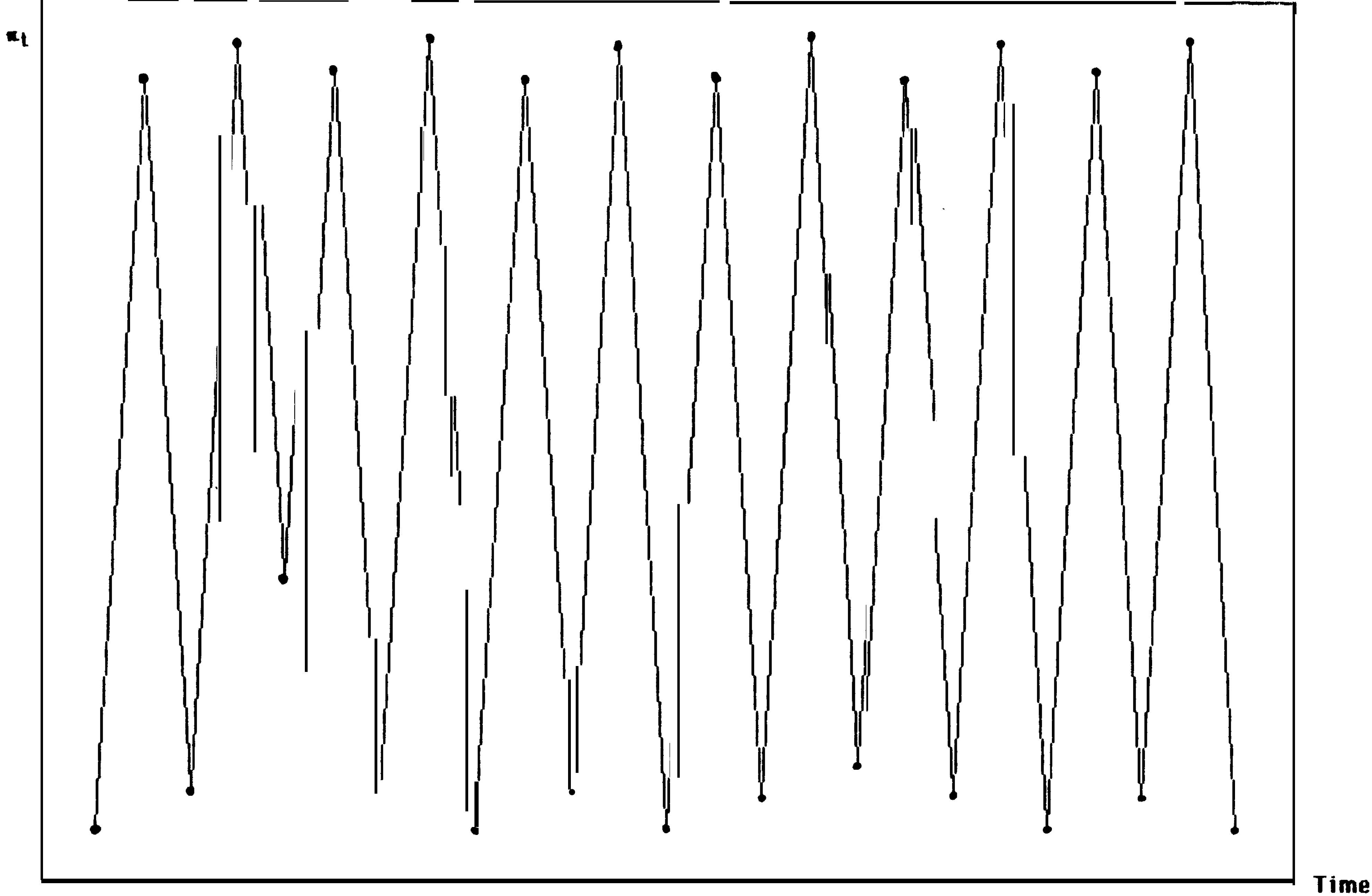
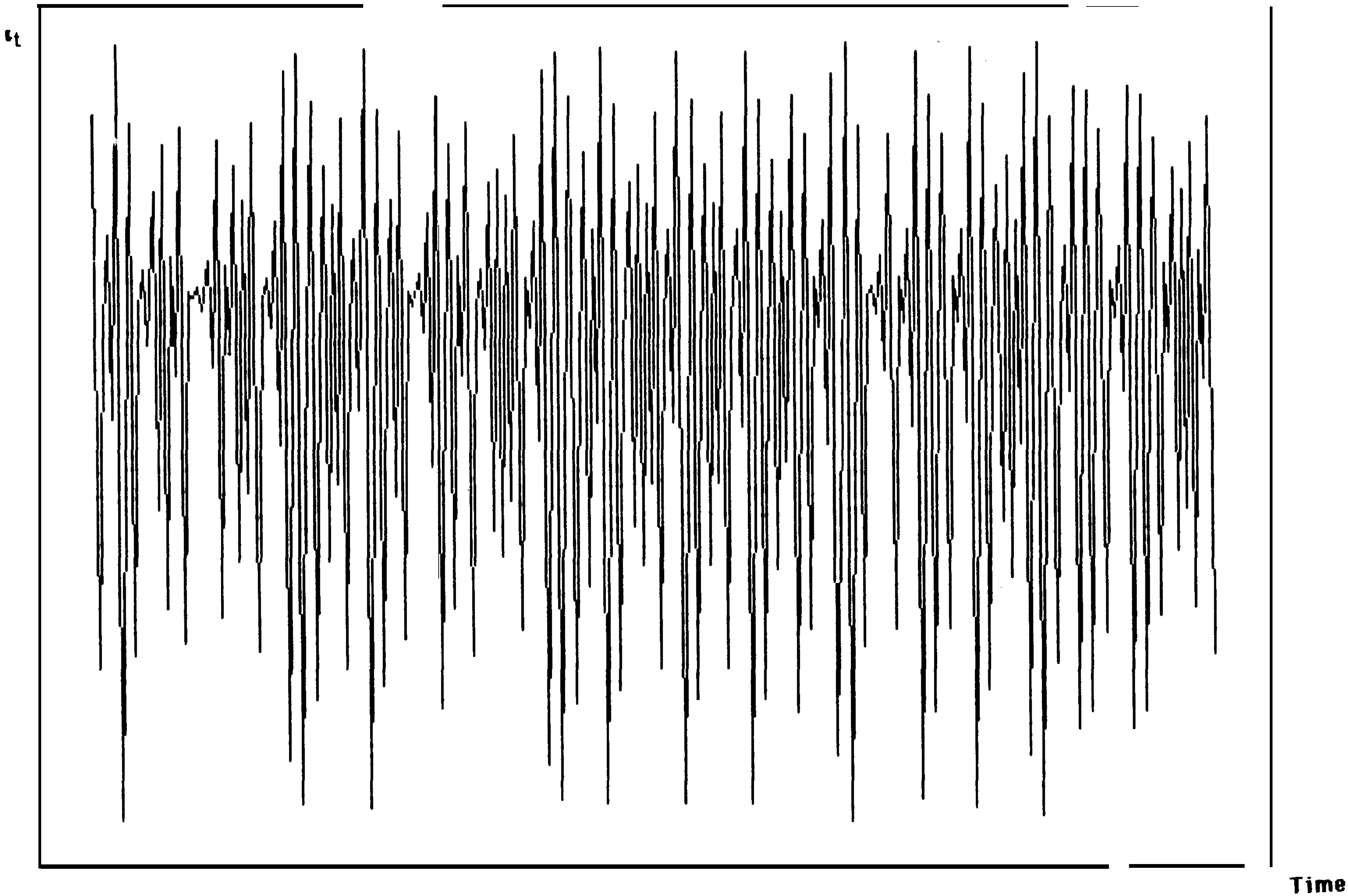


FIGURE 2



Simulation of (8), with γ = , ϕ = 1.05.

FIGURE 3

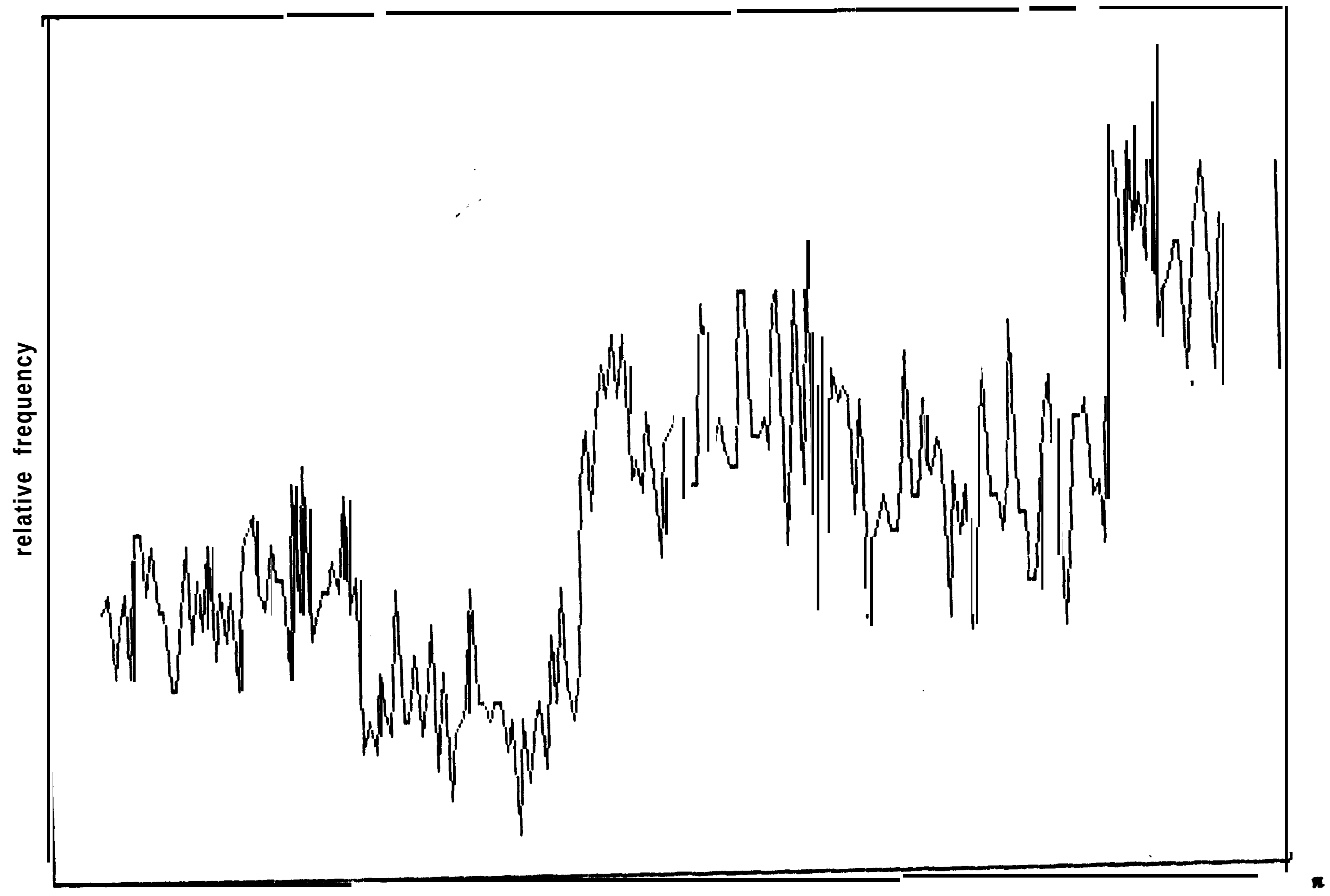


Simulation of (8), with $\gamma = 1, \phi = 2.1$

min π_t =.089
max π_t = .11

iterations = 500

FIGURE 4



min π_1 = .089
max π_1 = .11

min relative frequency = .001
max relative frequency = .0108

Histogram of simulation of (8), with γ = 1.1, ϕ = 2.

iterations = 5000
bins = 200

FIGURE 5

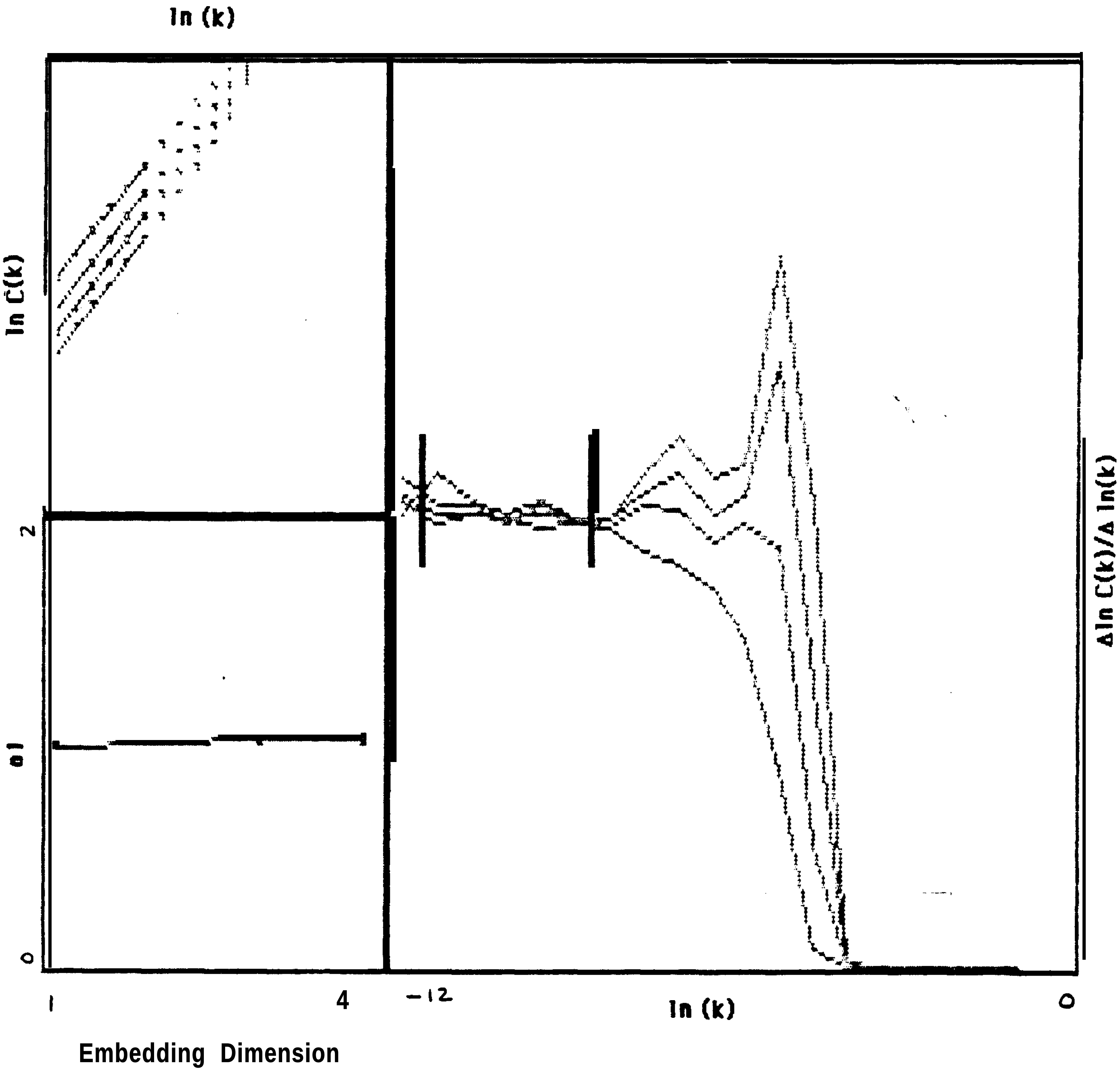


FIGURE 6

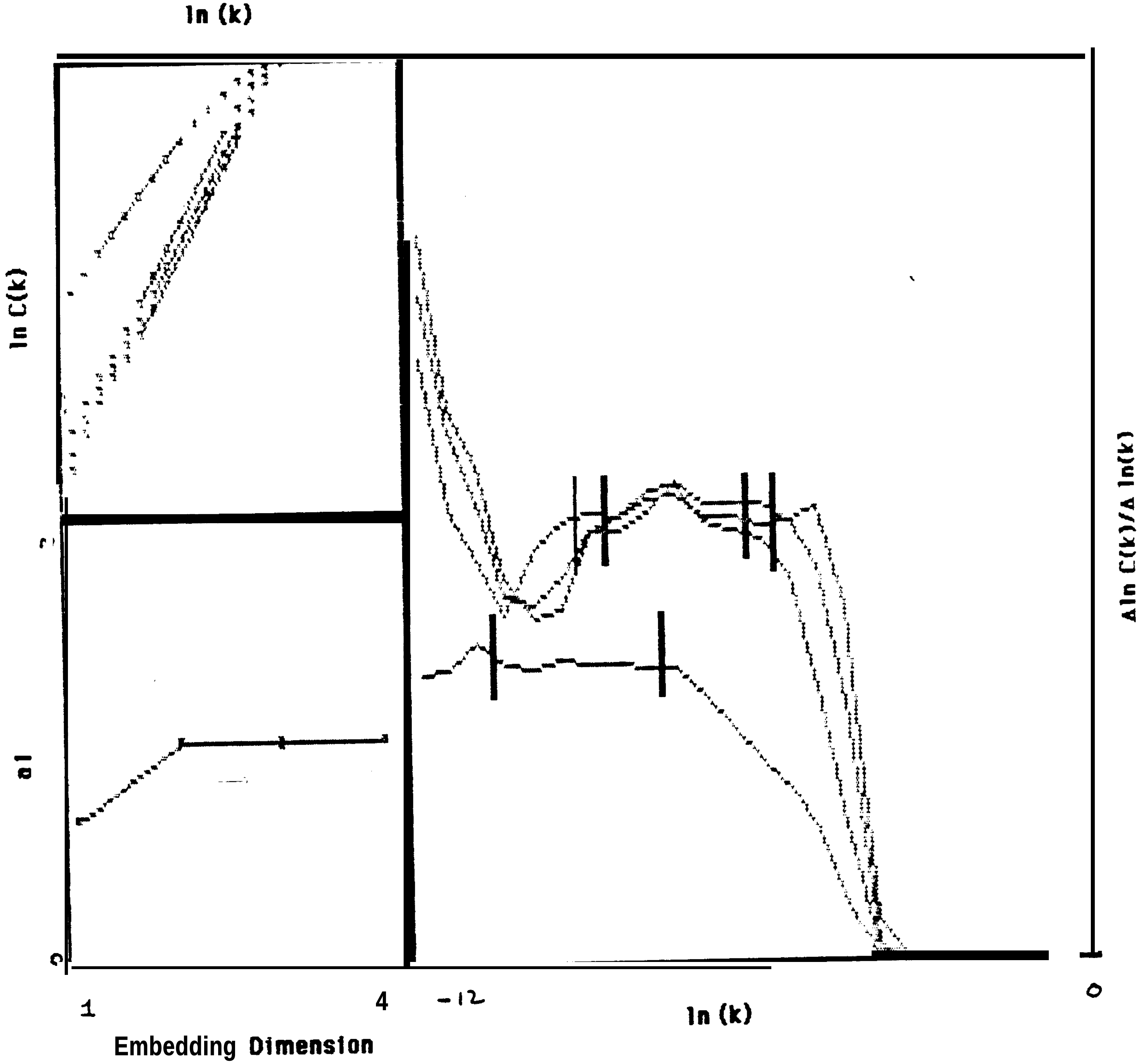
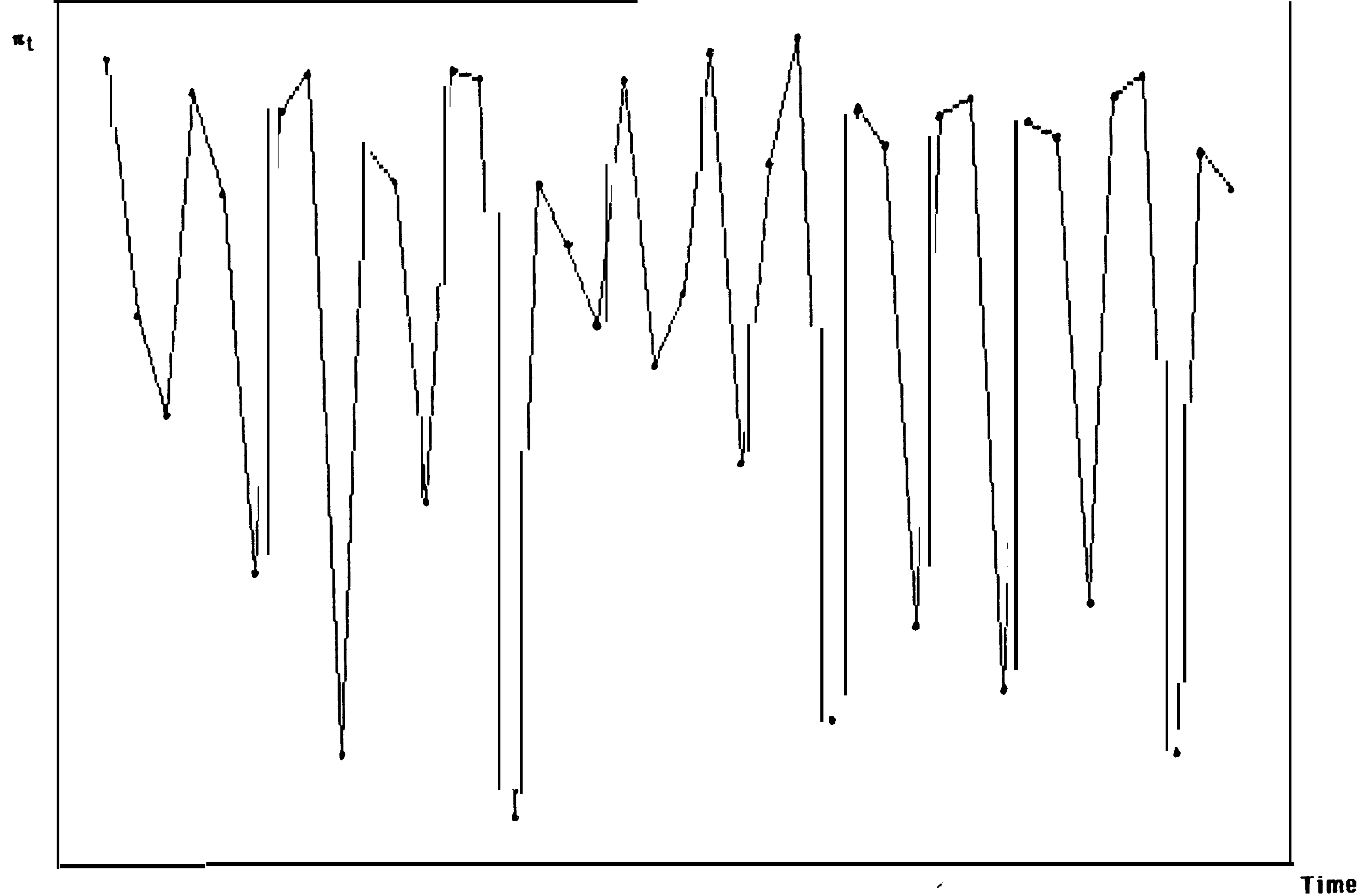


FIGURE 7



Simulation of (13), with $\gamma = 1.1$, $\phi = 3.5$

min π_t =.04
max π_t = .055

iterations = 40

FIGURE 8